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Quantum mechanics on the lightcone: II. the spin-$\frac{1}{2}$ case

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Abstract. We present (1) a set of spin-$\frac{1}{2}$ Poincaré group operators acting on the past lightcone, (2) a continuity equation resulting from the Hamiltonian with a positive-definite density, (3) the spin-$\frac{1}{2}$ basis states. In the non-relativistic limit our Hamiltonian is equivalent to the Pauli–Schrödinger equation. We also consider the Hamiltonian modified by a radial potential.

1. Introduction

We follow on from our paper on the spin-zero case [1]. We know of two previous papers on the spin-$\frac{1}{2}$ lightcone quantum theory. Peres [2] derived a set of spin-$\frac{1}{2}$ operators satisfying the Poincaré group algebra, and this paper may be regarded as an extension of Peres' work in that:

(1) We present an alternative set of operators related to his by a unitary transformation—this new set has the advantage that the Hamiltonian reduces to the Pauli–Schrödinger Hamiltonian in the non-relativistic limit.

(2) We define the inverse operators p^{-1} in the Peres paper.

(3) We derive a 4-current satisfying the continuity equation from the new Hamiltonian, the zero component of which is a positive-definite density.

(4) We find the basis states, which turn out to be those found by Derrick [3], from a different approach.

The spin-$\frac{1}{2}$ wavefunctions for both positive and negative energies (spin up and spin down) are two-component wavefunctions, instead of the usual Dirac four-component wavefunctions.

Finally in section 5 we apply the spin-$\frac{1}{2}$ Hamiltonian modified by a radial (retarded) potential along the past lightcone: the energy levels turn out to be the same as in the Dirac case with a corresponding radial potential, although the wavefunction is different.

2. The set of spin-$\frac{1}{2}$ operators in lightcone coordinates

In [1] we found a set of spin-zero operators in lightcone coordinates satisfying the the Poincaré group algebra:

$$[p^\lambda, p^\mu] = 0 \tag{2.1}$$

$$[j^{\lambda\mu}, p^\nu] = \mathrm{i}(\eta^{\mu\nu} p^\lambda - \eta^{\lambda\nu} p^\mu) \tag{2.2}$$

$$[j^{\lambda\mu}, j^{\nu\rho}] = \mathrm{i}(\eta^{\lambda\rho} j^{\mu\nu} + \eta^{\mu\nu} j^{\lambda\rho} - \eta^{\lambda\nu} j^{\mu\rho} - \eta^{\mu\rho} j^{\lambda\nu})\,. \tag{2.3}$$

In the non-relativistic limit, when the past lightcones get flattened out to the usual constant-time hyperplanes, the spin-zero operators reduce to the well known operators obeying the Galilei group algebra, with the usual Schrödinger Hamiltonian. Here we adapt the spin-zero operators to a set of spin-$\frac{1}{2}$ operators, satisfying the the Poincaré group algebra, and the spin-$\frac{1}{2}$ Hamiltonian reduces to the Pauli–Schrödinger Hamiltonian in the non-relativistic limit. We will use the results and notation of [1].

First we follow Peres [2] and Derrick [3] by adding the spin term $\frac{1}{2}\boldsymbol{\sigma}$ to the $\boldsymbol{J}$ rotation operators, and a $\frac{1}{2}(\boldsymbol{\sigma}\times\frac{\boldsymbol{y}}{y})$ term to the $\boldsymbol{K}$ boost operators, where $\boldsymbol{\sigma}$ are the usual Pauli matrices. So $\boldsymbol{J}$, $\boldsymbol{K}$ now become the 2×2 operators

$$\boldsymbol{J} \equiv (j^{23}, j^{31}, j^{12}) = -\mathrm{i}\boldsymbol{y}\times\frac{\partial}{\partial \boldsymbol{y}} + \frac{1}{2}\boldsymbol{\sigma} \tag{2.4}$$

$$\boldsymbol{K} \equiv (j^{10}, j^{20}, j^{30}) = -\mathrm{i}y\frac{\partial}{\partial \boldsymbol{y}} + \boldsymbol{y}m + \frac{1}{2}\left(\boldsymbol{\sigma}\times\frac{\boldsymbol{y}}{y}\right). \tag{2.5}$$

We will show that the following energy–momentum operators together with $\boldsymbol{J}$, $\boldsymbol{K}$ satisfy the Poincaré group algebra:

$$H_{\mathrm{spin}\,\frac{1}{2}} = m + \tfrac{1}{2}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})\sqrt{y}\,\Sigma_m^{-1}\sqrt{y}\,(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}}) \tag{2.6}$$

$$\boldsymbol{p}_{\mathrm{spin}\,\frac{1}{2}} = \hat{\boldsymbol{\pi}} - \tfrac{1}{2}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})\sqrt{y}\,\Sigma_m^{-1}\sqrt{y}\,\frac{\boldsymbol{y}}{y}\,(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}}) \tag{2.7}$$

where [1]

$$\hat{\boldsymbol{\pi}} \equiv -\mathrm{i}\sqrt{y}\,\frac{\partial}{\partial \boldsymbol{y}}\frac{1}{\sqrt{y}} \tag{2.8}$$

and Σ_m^{-1} is a Lorentz-invariant integral operator [1]. We can see that $H_{\mathrm{spin}\,\frac{1}{2}}$ is a symmetrized version of the 2×2 form of the spin-zero Hamiltonian H_{zero}, where (see (3.10) in [1]) $H_{\mathrm{zero}} \equiv m + \frac{1}{2}\Sigma_m^{-1}\sqrt{y}\,\hat{\boldsymbol{\pi}}^2\sqrt{y}$.

First note that by multiplying the spin-$\frac{1}{2}$ Hamiltonian operator (2.6) on the right by $(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})\sqrt{y}$ we obtain

$$\begin{aligned} H_{\mathrm{spin}\,\frac{1}{2}}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})\sqrt{y} &\equiv m(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})\sqrt{y} + \tfrac{1}{2}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})\sqrt{y}\,\Sigma_m^{-1}\sqrt{y}\,(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})^2\sqrt{y} \\ &= m(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})\sqrt{y} + \tfrac{1}{2}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})\sqrt{y}\,\Sigma_m^{-1}\sqrt{y}\,\hat{\boldsymbol{\pi}}^2\sqrt{y} \equiv (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})\sqrt{y}\,H_{\mathrm{zero}}. \end{aligned} \tag{2.9}$$

Similarly we can derive

$$\boldsymbol{p}_{\mathrm{spin}\,\frac{1}{2}}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})\sqrt{y} = (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})\sqrt{y}\,\boldsymbol{p}_{\mathrm{zero}} \tag{2.10}$$

where $\boldsymbol{p}_{\mathrm{spin}\,\frac{1}{2}}$ is the spin-zero momentum operator [1]. Then calling the operator $(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})\sqrt{y} \equiv \hat{O}$ and using (2.10), (2.11) we can deduce the following commutation relations:

$$\begin{aligned} [p^{\lambda}_{\mathrm{spin}\,\frac{1}{2}}, p^{\mu}_{\mathrm{spin}\,\frac{1}{2}}]\,\hat{O} &\equiv p^{\lambda}_{\mathrm{spin}\,\frac{1}{2}}\,p^{\mu}_{\mathrm{spin}\,\frac{1}{2}}\,\hat{O} - p^{\mu}_{\mathrm{spin}\,\frac{1}{2}}\,p^{\lambda}_{\mathrm{spin}\,\frac{1}{2}}\,\hat{O} \\ &= p^{\lambda}_{\mathrm{spin}\,\frac{1}{2}}\,\hat{O}\,p^{\mu}_{\mathrm{zero}} - p^{\mu}_{\mathrm{spin}\,\frac{1}{2}}\,\hat{O}\,p^{\lambda}_{\mathrm{zero}} = \hat{O}\,p^{\lambda}_{\mathrm{zero}}\,p^{\mu}_{\mathrm{zero}} - \hat{O}\,p^{\mu}_{\mathrm{zero}}\,p^{\lambda}_{\mathrm{zero}} \\ &\equiv \hat{O}\,[p^{\lambda}_{\mathrm{zero}}\,p^{\mu}_{\mathrm{zero}}] \end{aligned} \tag{2.11}$$

so that as the spin-zero energy–momentum operators commute it follows that the spin-$\frac{1}{2}$ operators also commute, satisfying (2.1).

Next we consider the 'mixed' commutation relations (2.2). The relations involving the boost operator $[\boldsymbol{K}, H] = \mathrm{i}\boldsymbol{p}$ and $[K^a, p^b] = \mathrm{i} H \delta^{ab}$, may be checked with the aid of the following identities.

Due to $(-y, \boldsymbol{y})$ being a 4-vector [1], it follows that

$$[K^a, y^b] = -\mathrm{i}y\,\delta^{ab} \qquad [\boldsymbol{K}, y] = -\mathrm{i}\boldsymbol{y} \tag{2.12}$$

and as Σ_m is Lorentz-invariant [1],

$$[\boldsymbol{K}, \Sigma_m] = [\boldsymbol{K}, \Sigma_m^{-1}] = 0\,. \tag{2.13}$$

Also needed are

$$[\boldsymbol{K}, (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})] = \mathrm{i}\,\boldsymbol{\sigma}\frac{1}{\sqrt{y}}\Sigma_m\frac{1}{\sqrt{y}} \tag{2.14}$$

$$[K^a, \hat{\pi}^b] = \frac{\mathrm{i}}{2}\left[\left(\sigma^a\frac{y^b}{y}\right)(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}}) + (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})\left(\sigma^a\frac{y^b}{y}\right)\right] + \mathrm{i}m\delta^{ab}\,. \tag{2.15}$$

The relations (2.2) involving the rotation operator $\boldsymbol{J}$ may be checked with the aid of $[\boldsymbol{J}, (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})] = 0$. The commutation identities of the Lorentz operators (2.3) are readily verified [2], and so finally the operators (2.4)–(2.7) are the required set of Poincaré group operators.

2.1. Hermiticity

The spin-zero Lorentz operators are Hermitian in the space $\mathcal{H}_y$ [1], which is the Lorentz-invariant positive-definite scalar product space on the past lightcone

$$\mathcal{H}_y : \quad \langle\phi|\,\psi\rangle_y \equiv \left\langle\phi\left|\frac{1}{y}\right|\psi\right\rangle \equiv \int \phi^*(\boldsymbol{y})\,\frac{1}{y}\,\psi(\boldsymbol{y})\,\mathrm{d}^3y \tag{2.16}$$

and the Hermiticity of the spin-$\frac{1}{2}$ Lorentz operators is unaffected by the extra spin terms. The Hamiltonian operator (2.6) is Hermitian in $\mathcal{H}_y$ due to its symmetric construction (recalling that $\hat{\boldsymbol{\pi}}$, y, Σ_m^{-1} are all Hermitian in $\mathcal{H}_y$ [1]). Then $\boldsymbol{p}$ must also be Hermitian in $\mathcal{H}_y$ due to the identity $\boldsymbol{p} = -\mathrm{i}[\boldsymbol{K}, H]$.

2.2. The evolution equation

The Hamiltonian (2.6) implies the evolution equation

$$\mathrm{i}\frac{\partial}{\partial T}\,\psi \equiv H_{\mathrm{spin}\,\frac{1}{2}}\,\psi = m\,\psi + \frac{1}{2}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})\sqrt{y}\,\Sigma_m^{-1}\sqrt{y}\,(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})\,\psi \tag{2.17}$$

which is an integro-differential equation of first order in time, where ψ is a two-component column vector. To determine the non-relativistic approximation of (2.17*a*), we temporarily reinsert the speed of light c (we have been using natural units with $\hbar = c = 1$) obtaining

$$\frac{H_{\mathrm{spin}\,\frac{1}{2}}}{c}\,\psi = mc\,\psi + \frac{1}{2}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})\sqrt{y}\,\Sigma_{mc}^{-1}\sqrt{y}\,(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})\,\psi \simeq mc\,\psi + \frac{1}{2}\frac{(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})^2}{mc}\,\psi \tag{2.17a}$$

in the non-relativistic limit when [1] $\Sigma_{mc}^{-1} \simeq 1/ymc$ as $c \to \infty$. Taking out the rest energy mc^2 and putting $\psi = \sqrt{y}\,\phi$, we obtain the non-relativistic Pauli–Schrödinger equation $H\,\phi = [(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})^2/2m]\,\phi$.

2.3. The Hamiltonian in radial form

From the Hamiltonian (2.6) we can derive

$$\begin{aligned} H_{\mathrm{spin}\frac{1}{2}} &= m + \frac{1}{2}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}})\sqrt{y}\,\Sigma_m^{-1}\sqrt{y}\,(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\pi}}) \\ &\equiv\ m + \frac{1}{2}\sqrt{y}\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right)\Sigma_m^{-1} y \left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right)\frac{1}{\sqrt{y}} \\ &= m + \frac{1}{2}\sqrt{y}\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right)\left(\boldsymbol{\sigma}\cdot\frac{\boldsymbol{y}}{y}\right)\Sigma_m^{-1} y\left(\boldsymbol{\sigma}\cdot\frac{\boldsymbol{y}}{y}\right)\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right)\frac{1}{\sqrt{y}} \end{aligned} \tag{2.18}$$

using $\left(\boldsymbol{\sigma}\cdot\frac{\boldsymbol{y}}{y}\right)^2 = 1$ and the fact that Σ_m^{-1} commutes with the angular variable $\left(\boldsymbol{\sigma}\cdot\frac{\boldsymbol{y}}{y}\right)$. In the appendix we prove that

$$\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right)\left(\boldsymbol{\sigma}\cdot\frac{\boldsymbol{y}}{y}\right) = \frac{1}{y}\Sigma_m - \frac{\mathrm{i}}{y}\hat{\kappa} - m$$

and

$$\left(\boldsymbol{\sigma}\cdot\frac{\boldsymbol{y}}{y}\right)\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right) = \frac{1}{y}\Sigma_m + \frac{\mathrm{i}}{y}\hat{\kappa} - m$$

where $\hat{\kappa}$ is the operator $(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{L}}+1)$ and $\hat{\boldsymbol{L}}$ is the angular momentum operator $-\mathrm{i}\boldsymbol{y}\times\frac{\partial}{\partial \boldsymbol{y}}$. So

$$\begin{aligned} H_{\mathrm{spin}\frac{1}{2}} &= m + \frac{1}{2}\sqrt{y}\left[\frac{1}{y}\Sigma_m - \frac{\mathrm{i}}{y}\hat{\kappa} - m\right]\Sigma_m^{-1} y\left[\frac{1}{y}\Sigma_m + \frac{\mathrm{i}}{y}\hat{\kappa} - m\right]\frac{1}{\sqrt{y}} \\ &= m + \frac{1}{2}\sqrt{y}\left[-\frac{\mathrm{i}}{y}\hat{\kappa} - m\right]\Sigma_m^{-1} y\left[\frac{\mathrm{i}}{y}\hat{\kappa} - m\right]\frac{1}{\sqrt{y}} \\ &\quad + \frac{1}{2}\sqrt{y}\left(\frac{\mathrm{i}}{y}\hat{\kappa} - m + \frac{1}{y}\Sigma_m - \frac{\mathrm{i}}{y}\hat{\kappa} - m\right)\frac{1}{\sqrt{y}} \\ &= \frac{1}{2}\frac{1}{\sqrt{y}}\Sigma_m\frac{1}{\sqrt{y}} + \frac{1}{2}\left[m + \frac{\mathrm{i}}{y}\hat{\kappa}\right]\sqrt{y}\,\Sigma_m^{-1}\sqrt{y}\left[m - \frac{\mathrm{i}}{y}\hat{\kappa}\right]. \end{aligned} \tag{2.19}$$

The operator $\hat{\kappa}$ is well known in Dirac theory. It is an angular operator so it commutes with $y\,\Sigma_m\,\Sigma_m^{-1}$. It also commutes with $H_{\mathrm{spin}\frac{1}{2}}$ as is easily seen from (2.19). It has positive and negative integer eigenvalues.

We note that Peres derived the Hamiltonian (his equation (43) in [2])

$$H_{\mathrm{Peres}} = \frac{1}{2}p + \frac{1}{2}\left[m + \frac{\mathrm{i}}{y}\hat{\kappa}\right]p^{-1}\left[m - \frac{\mathrm{i}}{y}\hat{\kappa}\right] \tag{2.20}$$

which was obtained by a very different approach (by reduction from the 4×4 Dirac operator in lightcone coordinates). His operator p is identical to our

$$\frac{\mathrm{e}^{\mathrm{i}my}}{\sqrt{y}}\Sigma_m\frac{\mathrm{e}^{-\mathrm{i}my}}{\sqrt{y}}$$

so naturally we can define his operator

$$p^{-1} = \mathrm{e}^{\mathrm{i}my}\sqrt{y}\,\Sigma_m^{-1}\sqrt{y}\,\mathrm{e}^{-\mathrm{i}my}\,.$$

As all other components in the Hamiltonians commute with $\mathrm{e}^{\mathrm{i}my}$, then the Hamiltonians can be made equivalent by the simple unitary transformation $H_{\mathrm{Peres}} = \mathrm{e}^{\mathrm{i}my}H_{\mathrm{spin}\frac{1}{2}}\mathrm{e}^{-\mathrm{i}my}$.

3. The continuity equation

The evolution equation with the Hamiltonian in the form (2.18) is

$$\mathrm{i}\frac{\partial}{\partial T}\psi \equiv H_{\mathrm{spin}\frac{1}{2}}\,\psi = m\,\psi + \frac{1}{2}\sqrt{y}\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right)\Sigma_m^{-1}y\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right)\frac{1}{\sqrt{y}}\,\psi \tag{3.1}$$

where ψ is a two-component column vector. We can write the above as the coupled equations

$$\mathrm{i}\frac{\partial}{\partial T}\psi \equiv m\,\psi + \frac{1}{2}\sqrt{y}\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right)\frac{1}{\sqrt{y}}\,\chi \tag{3.2a}$$

$$\chi = \sqrt{y}\,\Sigma_m^{-1}y\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right)\frac{1}{\sqrt{y}}\,\psi$$
$$\left(\text{equivalently}\quad \frac{1}{\sqrt{y}}\Sigma_m\frac{1}{\sqrt{y}}\chi = \sqrt{y}\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right)\frac{1}{\sqrt{y}}\,\psi\right) \tag{3.2b}$$

in which we regard χ as an auxillary variable which must satisfy (3.2*b*), whereas (3.2*a*) is the true dynamical equation. Next put

$$\psi = \sqrt{y}\,\phi_1 \qquad \chi = \sqrt{y}\,\phi_2 \tag{3.3}$$

and multiply (3.2) from the left by $1/\sqrt{y}$ obtaining

$$\mathrm{i}\frac{\partial}{\partial T}\phi_1 = m\,\phi_1 + \frac{1}{2}\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right)\phi_2 \tag{3.4a}$$

$$\frac{1}{y}\Sigma_m\,\phi_2 \equiv \left(-\frac{\mathrm{i}}{y}\frac{\partial}{\partial \boldsymbol{y}}\boldsymbol{y} + m\right)\phi_2 = \left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right)\phi_1\,. \tag{3.4b}$$

Equations (3.4) can now be used to derive a continuity equation as follows. First form the Hermitian conjugate equations to (3.4):

$$-\mathrm{i}\frac{\partial}{\partial T}\phi_1^\dagger = m\,\phi_1^\dagger + \frac{1}{2}\left(\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\phi_2^\dagger\right)\cdot\boldsymbol{\sigma} \tag{3.5a}$$

$$\left(\frac{\mathrm{i}}{y}\frac{\partial}{\partial \boldsymbol{y}}\boldsymbol{y} + m\right)\phi_2^\dagger = \left(\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\phi_1^\dagger\right)\cdot\boldsymbol{\sigma}\,. \tag{3.5b}$$

Now multiply equation (3.4*a*) from the left by $\phi_1^\dagger$, and (3.5*a*) from the right by ϕ_1, then subtract, obtaining

$$\begin{aligned}\mathrm{i}\left(\phi_1^\dagger\frac{\partial}{\partial T}\phi_1 + \left(\frac{\partial}{\partial T}\phi_1^\dagger\right)\phi_1\right) &= -\frac{\mathrm{i}}{2}\left[\phi_1^\dagger\boldsymbol{\sigma}\cdot\frac{\partial}{\partial \boldsymbol{y}}\phi_2 + \left(\frac{\partial}{\partial \boldsymbol{y}}\phi_2^\dagger\cdot\boldsymbol{\sigma}\right)\phi_1\right]\\ &= -\frac{\mathrm{i}}{2}\frac{\partial}{\partial \boldsymbol{y}}\cdot\left[\phi_1^\dagger\boldsymbol{\sigma}\,\phi_2 + \phi_2^\dagger\boldsymbol{\sigma}\,\phi_1\right] + \frac{\mathrm{i}}{2}\left[\left(\frac{\partial}{\partial \boldsymbol{y}}\phi_1^\dagger\cdot\boldsymbol{\sigma}\right)\phi_2 + \phi_2^\dagger\boldsymbol{\sigma}\cdot\frac{\partial}{\partial \boldsymbol{y}}\phi_1\right].\end{aligned}$$

We now rearrange the second term on the right-hand side using (3.4*b*) and (3.5*b*):

$$
\begin{aligned}
\mathrm{i}\left(\phi_1^\dagger \frac{\partial}{\partial T}\phi_1 + \left(\frac{\partial}{\partial T}\phi_1^\dagger\right)\phi_1\right) &= -\frac{\mathrm{i}}{2}\frac{\partial}{\partial \boldsymbol{y}}\cdot\left[\phi_1^\dagger \boldsymbol{\sigma}\,\phi_2 + \phi_2^\dagger \boldsymbol{\sigma}\,\phi_1\right] \\
&\quad + \frac{\mathrm{i}}{2}\left[\left(\frac{1}{y}\frac{\partial}{\partial y}y - \mathrm{i}m\right)\phi_2^\dagger\,\phi_2 + \phi_2^\dagger\left(\frac{1}{y}\frac{\partial}{\partial y}y + \mathrm{i}m\right)\phi_2\right] \\
&= -\frac{\mathrm{i}}{2}\frac{\partial}{\partial \boldsymbol{y}}\cdot\left[\phi_1^\dagger \boldsymbol{\sigma}\,\phi_2 + \phi_2^\dagger \boldsymbol{\sigma}\,\phi_1\right] + \frac{\mathrm{i}}{2}\frac{\partial}{\partial \boldsymbol{y}}\cdot\left[\phi_2^\dagger \frac{\boldsymbol{y}}{y}\phi_2\right]
\end{aligned}
\tag{3.6}
$$

where for the last line we have used the identity $\frac{1}{y}\frac{\partial}{\partial y}y = \frac{\partial}{\partial \boldsymbol{y}} \circ \frac{\boldsymbol{y}}{y}$.

Equation (3.6) is now in the form of a continuity equation $\frac{\partial}{\partial T}\rho^0 + \frac{\partial}{\partial \boldsymbol{y}}\cdot\boldsymbol{\rho} = 0$ where

$$
\rho^0 = \phi_1^\dagger\,\phi_1 \qquad \text{and} \qquad \boldsymbol{\rho} = \frac{1}{2}\left[\phi_1^\dagger \boldsymbol{\sigma}\,\phi_2 + \phi_2^\dagger \boldsymbol{\sigma}\,\phi_1 - \phi_2^\dagger \frac{\boldsymbol{y}}{y}\phi_2\right]. \tag{3.7}
$$

Recall from (3.3) that

$$
\phi_1 = \frac{1}{\sqrt{y}}\,\psi \qquad \phi_2 = \frac{1}{\sqrt{y}}\,\chi\,.
$$

So substituting the above into (3.7) we obtain finally

$$
\rho^0 = \psi^\dagger \frac{1}{y}\,\psi \qquad \text{and} \qquad \boldsymbol{\rho} = \frac{1}{2}\left[\psi^\dagger \frac{1}{y}\boldsymbol{\sigma}\,\chi + \chi^\dagger \boldsymbol{\sigma}\frac{1}{y}\,\psi - \chi^\dagger \frac{1}{y}\frac{\boldsymbol{y}}{y}\,\chi\right] \tag{3.8}
$$

where χ is (from (3.2*b*)) $\chi = \sqrt{y}\,\Sigma_m^{-1} y\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right)\frac{1}{\sqrt{y}}\,\psi$. We may regard the positive-definite ρ^0 as a probability density, and furthermore if we integrate this probability density over all space we obtain $\langle\psi\,|\,\psi\rangle_y$ where $\mathcal{H}_y$ is the positive-definite Lorentz-invariant scalar product space over the past lightcone (2.16).

4. Basis states

We will obtain positive and negative energy basis states of $H_{\mathrm{spin}\frac{1}{2}}$, deriving them from the spin-zero basis states [1].

Equation (2.9) implies that

$$
H_{\mathrm{spin}\frac{1}{2}}\sqrt{y}\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right)\phi = \sqrt{y}\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right)H_{\mathrm{zero}}\,\phi\,. \tag{4.1}
$$

Now the spin-zero Hamiltonian operator H_{zero} has the positive- and negative-energy scalar eigenfunctions [1] $u_{\boldsymbol{k}}$ and $v_{\boldsymbol{k}}$, which are $u_{\boldsymbol{k}} = \mathrm{e}^{\mathrm{i}\omega y - \mathrm{i}my + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}}$ and $v_{\boldsymbol{k}} = \mathrm{e}^{-\mathrm{i}\omega y - \mathrm{i}my - \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}}$ where $\omega \equiv \sqrt{m^2 + k^2}$. Then $H_{\mathrm{zero}}\,u_{\boldsymbol{k}} = \omega\,u_{\boldsymbol{k}}$ and $H_{\mathrm{zero}}\,u_{\boldsymbol{k}} = -\omega\,u_{\boldsymbol{k}}$. We can see from (4.1) how to construct eigenfunctions of

$H_{\mathrm{spin}\frac{1}{2}}$, having the form of 2×2 matrices—we can call them 'eigenmatrices'. They are $u'_{\boldsymbol{k}} = \sqrt{y}\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right) u_{\boldsymbol{k}}$ and $v'_{\boldsymbol{k}} = \sqrt{y}\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right) v_{\boldsymbol{k}}$, which are

$$u'_{\boldsymbol{k}} = \sqrt{y}\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right) u_{\boldsymbol{k}} \equiv \sqrt{y}\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right) \mathrm{e}^{\mathrm{i}\omega y - \mathrm{i}my + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}}$$

$$= \sqrt{y}\left[\boldsymbol{\sigma}\cdot(\omega - m)\frac{\boldsymbol{y}}{y} + \boldsymbol{\sigma}\cdot\boldsymbol{k}\right] \mathrm{e}^{\mathrm{i}\omega y - \mathrm{i}my - \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}} \tag{4.2}$$

$$v'_{\boldsymbol{k}} = \sqrt{y}\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right) v_{\boldsymbol{k}} \equiv \sqrt{y}\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right) \mathrm{e}^{-\mathrm{i}\omega y - \mathrm{i}my - \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}}$$

$$= -\sqrt{y}\left[\boldsymbol{\sigma}\cdot(\omega + m)\frac{\boldsymbol{y}}{y} + \boldsymbol{\sigma}\cdot\boldsymbol{k}\right] \mathrm{e}^{-\mathrm{i}\omega y - \mathrm{i}my - \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}} . \tag{4.3}$$

We will use $v'_{\boldsymbol{k}}$ as a negative-energy (anti-particle) eigenmatrix of $H_{\mathrm{spin}\frac{1}{2}}$, but $u'_{\boldsymbol{k}}$ is unsatisfactory since it is identically zero if the momenta $\boldsymbol{k}$ are zero.

We now obtain a satisfactory positive-energy eigenmatrix. Note that we can perform a unitary transformation on $H_{\mathrm{spin}\frac{1}{2}}$ which changes the sign of m in this operator, i.e.

$$H_{\mathrm{spin}\frac{1}{2}}\{m \to -m\} = \mathrm{e}^{2\mathrm{i}my}\left(\boldsymbol{\sigma}\cdot\frac{\boldsymbol{y}}{y}\right) H_{\mathrm{spin}\frac{1}{2}}\left(\boldsymbol{\sigma}\cdot\frac{\boldsymbol{y}}{y}\right) \mathrm{e}^{-2\mathrm{i}my} \equiv T^{\dagger} H_{\mathrm{spin}\frac{1}{2}}\, T \tag{4.4}$$

which identity we prove in the appendix. Now since $u'_{\boldsymbol{k}}$ is an eigenmatrix of $H_{\mathrm{spin}\frac{1}{2}}$ with eigenvalue ω

$$H_{\mathrm{spin}\frac{1}{2}}\, u'_{\boldsymbol{k}} \equiv H_{\mathrm{spin}\frac{1}{2}}\sqrt{y}\left[\boldsymbol{\sigma}\cdot(\omega - m)\frac{\boldsymbol{y}}{y} + \boldsymbol{\sigma}\cdot\boldsymbol{k}\right] \mathrm{e}^{\mathrm{i}\omega y - \mathrm{i}my + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}}$$

$$= \omega\sqrt{y}\left[\boldsymbol{\sigma}\cdot(\omega - m)\frac{\boldsymbol{y}}{y} + \boldsymbol{\sigma}\cdot\boldsymbol{k}\right] \mathrm{e}^{\mathrm{i}\omega y - \mathrm{i}my + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}} .$$

Changing the sign of m and using (4.4), then

$$T^{\dagger} H_{\mathrm{spin}\frac{1}{2}}\, T\sqrt{y}\left[\boldsymbol{\sigma}\cdot(\omega + m)\frac{\boldsymbol{y}}{y} + \boldsymbol{\sigma}\cdot\boldsymbol{k}\right] \mathrm{e}^{\mathrm{i}\omega y + \mathrm{i}my + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}}$$

$$= \omega\sqrt{y}\left[\boldsymbol{\sigma}\cdot(\omega + m)\frac{\boldsymbol{y}}{y} + \boldsymbol{\sigma}\cdot\boldsymbol{k}\right] \mathrm{e}^{\mathrm{i}\omega y + \mathrm{i}my + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}}$$

$$H_{\mathrm{spin}\frac{1}{2}}\, T\sqrt{y}\left[\boldsymbol{\sigma}\cdot(\omega + m)\frac{\boldsymbol{y}}{y} + \boldsymbol{\sigma}\cdot\boldsymbol{k}\right] \mathrm{e}^{\mathrm{i}\omega y + \mathrm{i}my + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}} \tag{4.5}$$

$$= \omega\, T\sqrt{y}\left[\boldsymbol{\sigma}\cdot(\omega + m)\frac{\boldsymbol{y}}{y} + \boldsymbol{\sigma}\cdot\boldsymbol{k}\right] \mathrm{e}^{\mathrm{i}\omega y + \mathrm{i}my + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}}$$

$$H_{\mathrm{spin}\frac{1}{2}}\, U_{\boldsymbol{k}} = \omega\, U_{\boldsymbol{k}}$$

where

$$U_{\boldsymbol{k}} \equiv T\sqrt{y}\left[\boldsymbol{\sigma}\cdot(\omega + m)\frac{\boldsymbol{y}}{y} + \boldsymbol{\sigma}\cdot\boldsymbol{k}\right] \mathrm{e}^{\mathrm{i}\omega y + \mathrm{i}my + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}}$$

$$= \sqrt{y}\left[(\omega + m) + \left(\boldsymbol{\sigma}\cdot\frac{\boldsymbol{y}}{y}\right)(\boldsymbol{\sigma}\cdot\boldsymbol{k})\right] \mathrm{e}^{\mathrm{i}\omega y - \mathrm{i}my + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}} \tag{4.6}$$

is a satisfactory positive-energy eigenmatrix of $H_{\text{spin}\frac{1}{2}}$.

Our eigenmatrices, $U_{\boldsymbol{k}}$ and $V_{\boldsymbol{k}} \equiv v'_{\boldsymbol{k}}$ from (4.3), are the same (except that we have the extra factor $\mathrm{e}^{-\mathrm{i}my}$) as the basis states found by Derrick in [3] from a very different approach. As both columns of $U_{\boldsymbol{k}}$ and $V_{\boldsymbol{k}}$ satisfy the eigenvalue equation, we follow Derrick's interpretation of these states, that each column of $U_{\boldsymbol{k}}$, $V_{\boldsymbol{k}}$ is a two-component column vector eigensolution of $H_{\text{spin}\frac{1}{2}}$—which may be identified as alternative spin states. So we have the left- and right-hand columns of $U_{\boldsymbol{k}}$, which we call $U^{\mathrm{L}}_{\boldsymbol{k}}$, $U^{\mathrm{R}}_{\boldsymbol{k}}$, as the particle states, and $V^{\mathrm{L}}_{\boldsymbol{k}}$, $V^{\mathrm{R}}_{\boldsymbol{k}}$ as the anti-particle states. Note that *each state is a two-component column vector.* We shall see that all four states are orthogonal.

Explicitly for zero-momentum solutions we see from (4.6) and (4.3) that

$$U^{\mathrm{L}}_{\boldsymbol{k}} = 2m\sqrt{y}\begin{pmatrix}1\\0\end{pmatrix} \qquad U^{\mathrm{R}}_{\boldsymbol{k}} = 2m\sqrt{y}\begin{pmatrix}0\\1\end{pmatrix} \tag{3.15}$$

and

$$V^{\mathrm{L}}_{\boldsymbol{k}} = -2m\frac{1}{\sqrt{y}}\begin{pmatrix}y_3\\y_1+\mathrm{i}y_2\end{pmatrix}\mathrm{e}^{-2\mathrm{i}my} \qquad V^{\mathrm{R}}_{\boldsymbol{k}} = -2m\frac{1}{\sqrt{y}}\begin{pmatrix}y_1-\mathrm{i}y_2\\-y_3\end{pmatrix}\mathrm{e}^{-2\mathrm{i}my}\,. \tag{3.16}$$

All states are clearly orthogonal in the rest frame and so are orthogonal in any frame.

5. The spin-$\frac{1}{2}$ lightcone equation with a radial potential

In the last section we found the eigensolutions of the free-particle Hamiltonian (2.6), which have a continuous spectrum of eigenvalues. We now consider the Hamiltonian with a scalar potential V added. This potential in the general case will be a function of y, T, and so is a retarded potential acting on the surface of the past lightcone T (instead of on the constant-time hyperplane t as in the usual case). We might expect that static potentials—independant of T—would have the same effect as the corresponding hyperplane potential, and indeed we find that for a static radial potential (function of y) the energy levels are the same as those obtained by inserting the corresponding hyperplane potential (function of r) into the Dirac equation. However the eigensolutions have an extra factor compared to the Dirac case.

For stationary states we now have

$$[H_{\text{spin}\frac{1}{2}} + V(y)]\psi = E\psi\,. \tag{5.1}$$

Recall $H_{\text{spin}\frac{1}{2}}$ in the form (2.19). Then since $\hat{\kappa}$ commutes with s,

$$\sqrt{y}\left[\frac{1}{y}\Sigma_m + \left(m+\frac{\mathrm{i}}{y}\hat{\kappa}\right)\Sigma_m^{-1}y\left(m-\frac{\mathrm{i}}{y}\hat{\kappa}\right) + 2V\right]\frac{1}{\sqrt{y}}\psi = 2E\,\psi \tag{5.2}$$

where ψ is a two-component column vector. As V is a function of s the operator $\hat{\kappa}$ still commutes with the modified Hamiltonian, having positive or negative integer eigenvalues $\kappa = \pm 1, \pm 2, \ldots$. So now we can separate out the radial part of (5.2) as a *single-component* (scalar) equation. Replacing $\hat{\kappa}$ with κ,

$$\sqrt{y}\left[\frac{1}{y}\Sigma_m + \left(m + \frac{\mathrm{i}}{y}\kappa\right)\Sigma_m^{-1} y \left(m - \frac{\mathrm{i}}{y}\kappa\right)\right]\frac{1}{\sqrt{y}}\,\psi = 2(E-V)\,\psi \tag{5.3}$$

or putting $\psi = \frac{1}{\sqrt{y}}\,\phi_1$ and multiplying from the left by $\sqrt{y}$

$$\Sigma_m \frac{1}{y}\phi_1 + \left(m + \mathrm{i}\frac{\kappa}{y}\right) s\Sigma_m^{-1}\left(m - \mathrm{i}\frac{\kappa}{y}\right)\phi_1 = 2(E-V)\,\phi_1 \,. \tag{5.4}$$

Now define

$$s\Sigma_m^{-1}\left(m - \mathrm{i}\frac{\kappa}{y}\right)\phi_1 = \phi_2 \Leftrightarrow \left(m - \mathrm{i}\frac{\kappa}{y}\right)\phi_1 = \Sigma_m \frac{1}{y}\phi_2 \,.$$

Combining this last result with (5.4) we obtain the coupled equations

$$\Sigma_m \frac{1}{y}\phi_1 + \left(m + \mathrm{i}\frac{\kappa}{y}\right)\phi_2 = 2(E-V)\,\phi_1$$
$$\Sigma_m \frac{1}{y}\phi_2 - \left(m - \mathrm{i}\frac{\kappa}{y}\right)\phi_1 = 0 \,.$$

We make these equations more symmetrical by making the transformations $\phi_1 = \mathrm{e}^{\mathrm{i}(E-m)y}\,\psi_1$, $\phi_2 = \mathrm{e}^{\mathrm{i}(E-m)y}\,\psi_2$, then recalling that $\Sigma_m \frac{1}{y} \equiv -\mathrm{i}\frac{\partial}{\partial y} + m \equiv -\mathrm{i}\partial_y + m$,

$$\begin{aligned} -\mathrm{i}\partial_y\psi_1 + \left(m + \mathrm{i}\frac{\kappa}{y}\right)\psi_2 &= (E - 2V)\,\psi_1 \\ -\mathrm{i}\partial_y\psi_2 - \left(m - \mathrm{i}\frac{\kappa}{y}\right)\psi_1 &= -E\psi_2 \,. \end{aligned} \tag{5.5}$$

Now add and subtract (5.5) to obtain

$$-\mathrm{i}\partial_y(\psi_1+\psi_2) - m(\psi_1-\psi_2) + \mathrm{i}\frac{\kappa}{y}(\psi_1+\psi_2) = E(\psi_1-\psi_2) - 2V\,\psi_1$$
$$-\mathrm{i}\partial_y(\psi_1-\psi_2) + m(\psi_1+\psi_2) - \mathrm{i}\frac{\kappa}{y}(\psi_1-\psi_2) = E(\psi_1+\psi_2) - 2V\,\psi_1$$

or putting $(\psi_1+\psi_2) = \mathrm{i}F$ and $(\psi_1-\psi_2) = G$, then

$$-\mathrm{i}\partial_y \mathrm{i}F - mG + \mathrm{i}\frac{\kappa}{y}\mathrm{i}F = E\,G - V\,(G + \mathrm{i}F)$$
$$-\mathrm{i}\partial_y G + m\mathrm{i}F - \mathrm{i}\frac{\kappa}{y}G = E\,\mathrm{i}F - V\,(G + \mathrm{i}F) \,.$$

Rearranging and multiplying the second equation by i, we obtain

$$\begin{aligned} \partial_y F - \frac{\kappa}{y}F &= (m+E)G - \mathrm{i}V(F - \mathrm{i}G) \\ \partial_y G + \frac{\kappa}{y}G &= (m-E)F + V(F - \mathrm{i}G) \,. \end{aligned} \tag{5.6}$$

These equations are now in a similar form to the coupled radial equations obtained in the standard reduction of the Dirac equation with a radial potential $V(r)$. For example in Schiff's [5] treatment of the Dirac radial equations he obtains (equation (53.15)), in our units,

$$\begin{aligned} \partial_r F' - \frac{\kappa}{r} F' &= (m+E)G' - V G' \\ \partial_r G' + \frac{\kappa}{r} G' &= (m-E)F' + V F' . \end{aligned} \tag{5.7}$$

If we multiply both equations (5.6) from the left by $\exp(\mathrm{i}\int V\mathrm{d}y)$ they take the form (5.7) with

$$F' = \exp\left(\mathrm{i}\int V\mathrm{d}y\right) F \qquad G' = \exp\left(\mathrm{i}\int V\mathrm{d}y\right) G . \tag{5.8}$$

Thus we obtain *the same energy levels* as for the Dirac case, which are dictated by boundary conditions for the radial wavefunctions, but our wavefunctions have the extra factor $\exp(-\mathrm{i}\int V\mathrm{d}y)$. In the case of the Coulomb potential $e^2/4\pi y$, this is $\exp[\mathrm{i}(e^2/4\pi)\log y]$, which represents a rapidly oscillating term as $y \to 0$. Of course when this term is multiplied by its complex conjugate it effectively disappears. Note that (5.6) may be regarded as equivalent in form to (5.7) but with an additional radial vector potential inserted which is equal to the original scalar potential. This might well be expected due to the change to lightcone coordinates. Then (5.8) may be regarded as a gauge transformation.

6. Concluding remarks

Although our Hamiltonian appears to be very different from Dirac's, the energy eigenvalues for a radial (retarded) potential turn out to be the same as in the Dirac case with an equivalent potential. Our wavefunctions are only two-valued instead of four-valued.

In view of the well known problems of relativistic position operators [6], we make the following comments on our $\boldsymbol{y}$ operator which, as it represents the particle position on the observer's past lightcone, we will call the retarded position operator:

(1) It is Hermitian in the positive-definite space $\mathcal{H}_y$.

(2) It is a covariant operator, part of the 4-vector $y \equiv (-y, \boldsymbol{y})$, in this respect different from the usual case, where the zero component of position is a parameter (c-number).

(3) As $\boldsymbol{y}$ is the retarded position, the apparent velocity $\boldsymbol{w} \equiv \frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}T}$, which is the actual velocity perceived by the observer, has very different properties from the usual velocity. From [1] (equations (2.10) and (2.11)) we know that

$$\boldsymbol{w} = \frac{y\boldsymbol{p}}{\boldsymbol{y}\cdot\boldsymbol{\pi} + ym} = \frac{y\boldsymbol{p}}{yH + \boldsymbol{y}\cdot\boldsymbol{p}} \equiv \frac{\boldsymbol{p}}{H + p^{\|}} \tag{6.1}$$

so that an extra radial momentum term $p^{\|} \equiv \frac{\boldsymbol{y}}{y}\cdot\boldsymbol{p}$ appears in the denominator. For simplicity now consider the case when the particle is directly approaching or receding, i.e. $|\boldsymbol{w}| = \pm w^{\|}$. Then

$$w^{\|} = \frac{p^{\|}}{H + p^{\|}} \tag{6.2}$$

and as $-H < p^{\|} < H$, (6.2) implies that $w^{\|}$ lies within the limits

$$-\infty < w^{\|} < \tfrac{1}{2}\,.$$

For massless particles this is replaced by

$$w^{\|} = -\infty \quad \text{(approaching)} \qquad w^{\|} = \tfrac{1}{2} \quad \text{(receding)}\,.$$

The negative infinite velocity applies to a directly approaching photon, which would not exist on the observer's stack of past lightcones until a certain time T, when it would immediately be present at the origin.

(4) The usual relativistic spin-$\frac{1}{2}$ Dirac Hamiltonian results in velocities with eigenvalues of $\pm c$—the well known Zitterbewegung effect. The Foldy–Wouthuysen [7] transformation yields the usual velocities $\frac{p}{H}$ but at the cost of making the position a non-local (integral) operator. Our operator is both local and has no oscillatory (Zitterbewegung-type) components.

The question of velocity in the usual relativistic quantum mechanics—from the point of view of the localization of evolving wavepackets—has been discussed in a previous paper of ours [8]. A similar analysis of position measurement in lightcone quantum mechanics will be discussed elsewhere.

Appendix

We prove the identities used in deriving (2.19), using the identity $(\boldsymbol{\sigma}\cdot\boldsymbol{m})\,(\boldsymbol{\sigma}\cdot\boldsymbol{n}) = (\boldsymbol{m}\cdot\boldsymbol{n}) + \mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{m}\times\boldsymbol{n})$ for any 3-vector operators $\boldsymbol{m}$, $\boldsymbol{n}$—provided $\boldsymbol{m}$ and $\boldsymbol{n}$ commute with $\boldsymbol{\sigma}$.

$$\begin{aligned}\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right)\left(\boldsymbol{\sigma}\cdot\frac{\boldsymbol{y}}{y}\right) &= \left(-\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\cdot\frac{\boldsymbol{y}}{y}\right) + \mathrm{i}\boldsymbol{\sigma}\cdot\left(-\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\times\frac{\boldsymbol{y}}{y}\right) = -\mathrm{i}\left(\frac{\partial}{\partial y}+\frac{2}{y}\right) + \mathrm{i}\boldsymbol{\sigma}\cdot\left(-\frac{\hat{\boldsymbol{L}}}{y}\right) \\ &= -\frac{\mathrm{i}}{y}\frac{\partial}{\partial y}y - \frac{\mathrm{i}}{y}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{L}}+1) = \frac{1}{y}\Sigma_m - m - \frac{\mathrm{i}}{y}\hat{\kappa} \end{aligned} \tag{A1}$$

where $\hat{\kappa}$ is the operator $(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{L}}+1)$. Similarly

$$\begin{aligned}\left(\boldsymbol{\sigma}\cdot\frac{\boldsymbol{y}}{y}\right)\left(\boldsymbol{\sigma}\cdot -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right) &= -\mathrm{i}\frac{\boldsymbol{y}}{y}\cdot\frac{\partial}{\partial \boldsymbol{y}} + \mathrm{i}\boldsymbol{\sigma}\cdot\left(\frac{\boldsymbol{y}}{y}\times -\mathrm{i}\frac{\partial}{\partial \boldsymbol{y}}\right) = -\mathrm{i}\frac{\partial}{\partial y} + \mathrm{i}\boldsymbol{\sigma}\cdot\left(\frac{\hat{\boldsymbol{L}}}{y}\right) \\ &= \frac{1}{y}\Sigma_m - m + \frac{\mathrm{i}}{y}\hat{\kappa}\,. \end{aligned} \tag{A2}$$

It is known (e.g. p 354 in [9]) that $\left(\boldsymbol{\sigma}\cdot\frac{\boldsymbol{y}}{y}\right)$ anti-commutes with $\hat{\kappa}$, i.e.

$$\hat{\kappa}\left(\boldsymbol{\sigma}\cdot\frac{\boldsymbol{y}}{y}\right) + \left(\boldsymbol{\sigma}\cdot\frac{\boldsymbol{y}}{y}\right)\hat{\kappa} = 0 \qquad \text{or} \qquad \left(\boldsymbol{\sigma}\cdot\frac{\boldsymbol{y}}{y}\right)\hat{\kappa}\left(\boldsymbol{\sigma}\cdot\frac{\boldsymbol{y}}{y}\right) = -\hat{\kappa}\,. \tag{A3}$$

So with $H_{\mathrm{spin}\,\frac{1}{2}}$ in the radial form of (2.19)

$$H_{\mathrm{spin}\,\frac{1}{2}} = \frac{1}{2}\,\frac{1}{\sqrt{y}}\Sigma_m\frac{1}{\sqrt{y}} + \frac{1}{2}\left[m + \frac{\mathrm{i}}{y}\hat{\kappa}\right]\sqrt{y}\,\Sigma_m^{-1}\sqrt{y}\,\left[m - \frac{\mathrm{i}}{y}\hat{\kappa}\right]$$

and recalling that $\left(\sigma\cdot\frac{y}{y}\right)$ commutes with Σ_m, Σ_m^{-1}, then

$$\left(\sigma\cdot\frac{y}{y}\right) H_{\mathrm{spin}\,\frac{1}{2}} \left(\sigma\cdot\frac{y}{y}\right) \equiv \left(\sigma\cdot\frac{y}{y}\right)\left[\frac{1}{2}\frac{1}{\sqrt{y}}\Sigma_m\frac{1}{\sqrt{y}} + \frac{1}{2}\left(m+\frac{\mathrm{i}}{y}\hat{\kappa}\right)\sqrt{y}\,\Sigma_m^{-1}\sqrt{y}\left(m-\frac{\mathrm{i}}{y}\hat{\kappa}\right)\right]\left(\sigma\cdot\frac{y}{y}\right)$$

$$= \frac{1}{2}\frac{1}{\sqrt{y}}\Sigma_m\frac{1}{\sqrt{y}} + \frac{1}{2}\left(m-\frac{\mathrm{i}}{y}\hat{\kappa}\right)\sqrt{y}\,\Sigma_m^{-1}\sqrt{y}\left(m+\frac{\mathrm{i}}{y}\hat{\kappa}\right)$$

$$= \frac{1}{2}\frac{1}{\sqrt{y}}\Sigma_m\frac{1}{\sqrt{y}} + \frac{1}{2}\left(-m+\frac{\mathrm{i}}{y}\hat{\kappa}\right)\sqrt{y}\,\Sigma_m^{-1}\sqrt{y}\left(-m-\frac{\mathrm{i}}{y}\hat{\kappa}\right). \qquad \text{(A4)}$$

Now use the fact that $\mathrm{e}^{2\mathrm{i}my}\,\Sigma_m\,\mathrm{e}^{-2\mathrm{i}my} = \Sigma_{(-m)}$, so $\mathrm{e}^{2\mathrm{i}my}\,\Sigma_m^{-1}\,\mathrm{e}^{-2\mathrm{i}my} = \Sigma_{(-m)}^{-1}$, then

$$\mathrm{e}^{2\mathrm{i}my}\left(\sigma\cdot\frac{y}{y}\right) H_{\mathrm{spin}\,\frac{1}{2}} \left(\sigma\cdot\frac{y}{y}\right)\mathrm{e}^{-2\mathrm{i}my}$$

$$= \frac{1}{2}\frac{1}{\sqrt{y}}\Sigma_{(-m)}\frac{1}{\sqrt{y}} + \frac{1}{2}\left(-m+\frac{\mathrm{i}}{y}\hat{\kappa}\right)\sqrt{y}\Sigma_{(-m)}^{-1}\sqrt{y}\left(-m-\frac{\mathrm{i}}{y}\hat{\kappa}\right)$$

$$\equiv H_{\mathrm{spin}\,\frac{1}{2}} : \{m \to -m\} \qquad \text{(A5)}$$

which is (4.4).

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